

A new Z_3 -graded quantum group

Salih Celik ¹

Department of Mathematics, Yildiz Technical University,
DAVUTPASA-Esenler, Istanbul, 34210 TURKEY.

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Abstract

We introduce a Z_3 -graded version of exterior (Grassmann) algebra with two generators and using this object we obtain a new Z_3 -graded quantum group denoted by $O(\widetilde{GL}_q(2))$. We also discuss some properties of $O(\widetilde{GL}_q(2))$.

1 Introduction

Quantum plane [1] is a well known example in quantum group theory. One specific approach to represent a quantum group is to introduce quantum plane (and its dual). When there exists an appropriate set of noncommuting variables spanning linearly a representation space, the endomorphisms on that space preserving the noncommutative structure allows to set up a quantum group. The natural extension to Z_2 -graded space was introduced in [2]. The present work starts a Z_3 -graded version of the exterior plane, denoted by $\tilde{\mathbb{R}}_q^{0|2}$, where q is a cubic root of unity. In this case, of course, it will not go back to the original objects. The term "plane" is used as a formal title based upon its construction. Following the approach of the Manin's to quantum group $GL(2)$ we see that there exists a Z_3 -graded (quantum) group acting on the Z_3 -graded exterior plane. A detailed discussion of this group are given in Sect. 3. In [3] Chung finds commutation relations between the elements of a Z_3 -graded quantum 2×2 matrix using the differential schema established on quantum $(1+1)$ -superplane. With a similar idea, in [4] the author obtains similar (but not all the same) relations. However, all structures introduced in the present study are completely different from both [3] and [4] except for matrix T .

2 Z_3 -graded planes

The aim of this section is to introduce the Z_3 -graded version of the exterior algebra and its dual. It is known that the Manin's quantum plane is introduced as a q -deformation of commutative plane in the sense that it becomes the classical plane when q is equal to 1. In our case, the parameter q is a cubic root of unity and there is no return. To understand what this means, let's begin with recalling some facts about the exterior algebra.

¹E-mail: sacelik@yildiz.edu.tr

2.1 Z_3 -gradation

A Z_3 -graded vector space is a vector space V together with a decomposition $V = V_0 \oplus V_1 \oplus V_2$. Members of $V_0 \oplus V_1 \oplus V_2$ are called homogeneous elements. The *grade* (or *degree*) of a homogenous element $v \in V_i$ is denoted by $\tau(v) = i$, $i \in Z_3$. An element in V_0 (resp. V_1 and V_2) is of degree 0 (resp. 1 and 2).

A Z_3 -graded algebra A is a Z_3 -graded vector space $A = A_0 \oplus A_1 \oplus A_2$ which is also an associative algebra such that $A_i \cdot A_j \subset A_{i+j}$ or, equivalently, $\tau(\xi_1 \cdot \xi_2) = \tau(\xi_1) + \tau(\xi_2)$ for all homogeneous elements $\xi_1, \xi_2 \in A$.

2.2 The algebra of functions on the Z_3 -graded exterior plane

A possible way to generalize the Z_3 -graded exterior plane is to increase the power of nilpotency of its generators and to impose a Z_3 -graded commutation relation on the generators. We will assume that q is a cubic root of unity.

It is needed to put the wedge product between the coordinates of exterior plane, but it does not matter in the Z_3 -graded case.

Definition 2.1 Let $O(\tilde{\mathbb{R}}_q^{0|2})$ be the algebra with the generators θ and φ obeying the relations

$$\theta \cdot \varphi = q^2 \varphi \cdot \theta, \quad \theta^3 = 0 = \varphi^3 \quad (1)$$

where the coordinates θ and φ are of grade 1 and 2, respectively. We call $O(\tilde{\mathbb{R}}_q^{0|2})$ the algebra of functions on the Z_3 -graded exterior plane $\tilde{\mathbb{R}}_q^{0|2}$.

Definition 2.2 The Z_3 -graded plane $\tilde{\mathbb{R}}_q^{*0|2}$ with the function algebra

$$O(\tilde{\mathbb{R}}_q^{*0|2}) = K\{\xi, x\} / \langle \xi x - x \xi \rangle$$

is called Z_3 -graded dual exterior plane where the generators ξ, x are of degree 2, 0, respectively.

Hence, in accordance with Definition 2.2, we have

$$\tilde{\mathbb{R}}_q^{*0|2} \ni \begin{pmatrix} \xi \\ x \end{pmatrix} \iff \xi x = x \xi. \quad (2)$$

3 The Z_3 -graded (quantum) group

The algebraic group $SL(2, \mathbb{C})$ has coordinate algebra $O(SL(2, \mathbb{C}))$. This algebra is the quotient of the commutative polynomial algebra $\mathbb{C}[a, b, c, d]$ by the two-sided ideal generated by the element $ad - bc - 1$ where the indeterminates a, b, c, d are the coordinate functions on $SL(2, \mathbb{C})$. Using the group structure in $SL(2, \mathbb{C})$, we can encode it in terms of maps m (multiplication), η (identity) and S (inversion). Dualizing these maps to $O(SL(2, \mathbb{C}))$, we get the corresponding co-maps called comultiplication Δ , counit ϵ , and antipode S , respectively. The

axioms for the group structure of $SL(2, \mathbb{C})$, in terms of the maps, are then reversed giving us relations among the co-maps. The natural axioms satisfied in $O(SL(2, \mathbb{C}))$ by the maps $m, \eta, \Delta, \epsilon$ and S , it makes a Hopf algebra. The quantum group $O_q(SL(2, \mathbb{C}))$ is a noncommutative deformation of $O(SL(2, \mathbb{C}))$. General concepts related to quantum groups (Hopf algebras) can be found in the books of Klimyk and Schmüdgen [5] or Majid [6].

In this section, we will consider the 2×2 matrices acting on the Z_3 -graded exterior plane and will discuss the properties of such matrices. So, let a, β, γ, d be elements of an algebra A where the generators a and d are of degree 0, the generators γ and β are of degree 1 and 2, respectively. Let $\tilde{M}(2)$ be defined as the polynomial algebra $k[a, \beta, \gamma, d]$. It will sometimes be convenient and more illustrative to write a point (a, β, γ, d) of $\tilde{M}(2)$ in the matrix form

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = (t_{ij}). \quad (3)$$

We constitute the Z_3 -graded matrix algebra $\tilde{M}(2)$ as follows: We divided the algebra $\tilde{M}(2)$ into three parts in form $\tilde{M}(2) = A_{\bar{0}} \oplus A_{\bar{1}} \oplus A_{\bar{2}}$. In this case, if a matrix has the form of

$$T_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad (\text{resp.} \quad T_1 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}),$$

then it is an element of $A_{\bar{0}}$ (resp. $A_{\bar{1}}, A_{\bar{2}}$) and is of grade 0 (resp. 1, 2). This gives a Z_3 -graded structure to the algebra of matrices, in the sense that $\tau(T_i T_j) = \tau(T_i) + \tau(T_j) \pmod{3}$. It is easy to check that the product of two Z_3 -graded matrices is also a Z_3 -graded matrix. As it can easily be shown, matrices of the form (3) form a group provided that $ad - \beta\gamma \neq 0$. We denote this group by $\widetilde{GL}(2)$.

3.1 The algebra $O(\tilde{M}_q(2))$

To determine a q -analogue of the algebra $O(\tilde{M}(2))$, we will first obtain the commutation relations between the matrix elements of the matrix T .

If A and B are Z_3 -graded algebras, then their tensor product $A \otimes B$ is the Z_3 -graded algebra whose underlying space is Z_3 -graded tensor product of A and B . The following definition [7] gives the product rule for tensor product of algebras.

Definition 3.1 *If A is a Z_3 -graded algebra, then the product rule in the Z_3 -graded algebra $A \otimes A$ is defined by*

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = q^{\tau(a_2)\tau(a_3)} a_1 a_3 \otimes a_2 a_4 \quad (4)$$

where a_i 's are homogeneous elements in the algebra A .

Remark 1. It is well known that, the matrix T given in (3) defines the linear transformation $T : \tilde{\mathbb{R}}_q^{0|2} \longrightarrow \tilde{\mathbb{R}}_q^{0|2}$ and $T : \tilde{\mathbb{R}}_q^{*0|2} \longrightarrow \tilde{\mathbb{R}}_q^{*0|2}$. As a result of

these, we have $T\Theta = \Theta' \in \tilde{\mathbb{R}}_q^{0|2}$ and $T\Phi = \Phi' \in \tilde{\mathbb{R}}_q^{*0|2}$, where $\Theta = (\theta, \varphi)^t$ and $\Phi = (\xi, x)^t$. However, the relation $\alpha_1\alpha_2 = q^{\tau(\alpha_1)\tau(\alpha_2)}\alpha_2\alpha_1$ for all elements α_1 and α_2 in the Z_3 -graded algebra is inconsistent. Therefore, we will use the following transform while getting the commutation relations between the matrix elements of T .

Let a, β, γ, d be elements of the algebra $O(\tilde{M}(2))$. We also assume that the generators a and d are of degree 0, the generators γ and β are of degree 1 and 2, respectively. Then we can change the coordinates of a vector in $\tilde{\mathbb{R}}_q^{0|2}$ as follows

$$\Theta' = \begin{pmatrix} \theta' \\ \varphi' \end{pmatrix} := \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}, \quad \Theta'' = \begin{pmatrix} \theta'' \\ \varphi'' \end{pmatrix} := \begin{pmatrix} \theta & \varphi \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}. \quad (5)$$

So, we can give the following proposition that can be proved with straightforward computations.

Proposition 3.2 *The coordinates of Θ' and Θ'' satisfy (1) if and only if the generators a, β, γ, d fulfill the relations*

$$a\beta = \beta a, \quad \beta\gamma = \gamma\beta, \quad d\beta = \beta d, \quad (6)$$

$$a\gamma = q\gamma a, \quad d\gamma = q^2\gamma d, \quad (7)$$

$$ad = da + (q-1)\beta\gamma, \quad (8)$$

where q is a cubic root of unity.

Remark 2. Unlike the usual quantum group [1], one interesting feature is that the element β belongs to the center of the algebra.

Definition 3.3 *The Z_3 -graded algebra $O(\tilde{M}_q(2))$ is the quotient of the free algebra $k\{a, \beta, \gamma, d\}$ by the two-sided ideal J_q generated by the six relations (6)-(8) of Proposition 3.2.*

By relation (8), we have

$$D_q := ad - q\beta\gamma = da - \beta\gamma. \quad (9)$$

This element of $O(\tilde{M}_q(2))$ is called the Z_3 -graded determinant.

The proof of the following assertion is given by direct computation using the relations (6)-(8).

Remark 3. The Z_3 -graded quantum determinant defined in (9) commutes with a, β, γ and d , so that the requirement $D_q = 1$ is consistent.

Proposition 3.4 *Let T and T' be two matrices such that their matrix elements satisfy the relations (6)-(8). If all elements of T commute according to the rule (4) with all elements of T' , then the elements of the matrix (tensor) product TT' obey the relations (6)-(8). We also have*

$$D_q(T \dot{\otimes} T') = D_q(T) \otimes D_q(T').$$

Proof. Let the matrix (tensor) product of T with T' be

$$T \dot{\otimes} T' = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a' & \beta' \\ \gamma' & d' \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Then using the relations (6)-(8) with (4) we get

$$\begin{aligned} XY &= (a \otimes a' + \beta \otimes \gamma')(a \otimes \beta' + \beta \otimes d') \\ &= a^2 \otimes a' \beta' + a \beta \otimes a' d' + \beta a \otimes \gamma' \beta' + q^2 \beta^2 \otimes \gamma' d' \\ &= a^2 \otimes \beta' a' + a \beta \otimes d' a' + q a \beta \otimes \beta' \gamma' + \beta^2 \otimes d' \gamma' \\ YX &= (a \otimes \beta' + \beta \otimes d')(a \otimes a' + \beta \otimes \gamma') \\ &= a^2 \otimes \beta' a' + a \beta \otimes d' a' + q a \beta \otimes \beta' \gamma' + \beta^2 \otimes d' \gamma'. \end{aligned}$$

It can be similarly shown that relations $XZ = qZX$, $YZ = ZY$, etc., are provided. Proof of the latter as follows:

$$\begin{aligned} XW &= a \gamma \otimes a' \beta' + a d \otimes a' d' + q \beta \gamma \otimes \gamma' \beta' + \beta d \otimes \gamma' d' \\ YZ &= q^2 a \gamma \otimes \beta' a' + a d \otimes \beta' \gamma' + \beta \gamma \otimes d' a' + \beta d \otimes d' \gamma' \\ XW - qYZ &= a d \otimes (a' d' - q \beta' \gamma') - q \beta \gamma \otimes (d' a' - \gamma' \beta') \end{aligned}$$

and so $D_q(T \dot{\otimes} T')$ reduces to $D_q(T) \otimes D_q(T')$. \square

3.2 Bialgebra structure on $\tilde{M}_q(2)$

We now supply the algebra $O(\tilde{M}_q(2))$ with a bialgebra structure. The comultiplication and the counit will be the same as the usual quantum groups.

Proposition 3.5 (1) *There exist Z_3 -graded algebra homomorphisms*

$$\Delta : O(\tilde{M}_q(2)) \longrightarrow O(\tilde{M}_q(2)) \otimes O(\tilde{M}_q(2)), \quad \epsilon : O(\tilde{M}_q(2)) \longrightarrow \mathbb{C}$$

uniquely determined by

$$\Delta(a) = a \otimes a + \beta \otimes \gamma, \quad \Delta(\beta) = a \otimes \beta + \beta \otimes d, \quad (10)$$

$$\Delta(\gamma) = \gamma \otimes a + d \otimes \gamma, \quad \Delta(d) = \gamma \otimes \beta + d \otimes d, \quad (11)$$

$$\epsilon(a) = 1 = \epsilon(d), \quad \epsilon(\beta) = 0 = \epsilon(\gamma). \quad (12)$$

(2) *With these maps, the algebra $O(\tilde{M}_q(2))$ is a bialgebra which is neither commutative nor cocommutative.*

(3) *The quantum determinant D_q is group-like element of $O(\tilde{M}_q(2))$.*

Proof. (1) In order to prove that Δ and ϵ are algebra homomorphisms, it is enough to show that the relations (6)-(8) remain invariant under Δ and ϵ . As an sample let us show that $\Delta(a\beta) = \Delta(\beta a)$:

$$\begin{aligned} \Delta(a\beta) &= \Delta(a)\Delta(\beta) = (a \otimes a + \beta \otimes \gamma)(a \otimes \beta + \beta \otimes d) \\ &= a^2 \otimes a\beta + a\beta \otimes ad + \beta a \otimes \gamma\beta + q^2 \beta^2 \otimes \gamma d \\ &= a^2 \otimes \beta a + \beta a \otimes da + q a \beta \otimes \beta \gamma + \beta^2 \otimes d\gamma \\ \Delta(\beta a) &= \Delta(\beta)\Delta(a) = (a \otimes \beta + \beta \otimes d)(a \otimes a + \beta \otimes \gamma) \\ &= a^2 \otimes \beta a + q a \beta \otimes \beta \gamma + \beta a \otimes da + \beta^2 \otimes d\gamma. \end{aligned}$$

Analogously, one can prove another relations. For ϵ it is completely analogous.
(2) It is not difficult to check that the comultiplication Δ is coassociative in the sense that

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad (13)$$

and the counit ϵ has the property

$$m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta. \quad (14)$$

It follows that $O(\tilde{M}_q(2))$ is indeed a bialgebra.

(3) To prove that the Z_3 -graded determinant D_q is group-like, it is enough to show that

$$\Delta(D_q) = D_q \otimes D_q \quad \text{and} \quad \epsilon(D_q) = 1. \quad (15)$$

Indeed, some computations give

$$\begin{aligned} \Delta(D_q) &= \Delta(a)\Delta(d) - q\Delta(\beta)\Delta(\gamma) \\ &= ad \otimes ad + q\beta\gamma \otimes \beta\gamma - qad \otimes \beta\gamma - q\beta\gamma \otimes da \\ &= ad \otimes (ad - q\beta\gamma) + q\beta\gamma \otimes (\beta\gamma - da) \\ &= (ad - q\beta\gamma) \otimes (da - \beta\gamma) \end{aligned}$$

and $\epsilon(ad - q\beta\gamma) = \epsilon(a)\epsilon(d) - q\epsilon(\beta)\epsilon(\gamma) = 1$. \square

The bialgebra $O(\tilde{M}_q(2))$ is called the coordinate algebra of the Z_3 -graded (quantum) matrix space $\tilde{M}_q(2)$.

3.3 The Z_3 -graded Hopf algebra $O(\widetilde{GL}_q(2))$

Using the quantum determinant D_q belonging to the algebra $O(\tilde{M}_q(2))$, we can define a *new* Hopf algebra adding an inverse t^{-1} to $O(\tilde{M}_q(2))$. Let $O(\widetilde{GL}_q(2))$ be the quotient of the algebra $O(\tilde{M}_q(2))$ by the two-sided ideal generated by the element $tD_q - 1$. For short we write

$$O(\widetilde{GL}_q(2)) := O(\tilde{M}_q(2))[t]/\langle tD_q - 1 \rangle.$$

Then the algebra $O(\widetilde{GL}_q(2))$ is again a bialgebra.

Lemma 3.6 *The elements of the matrix*

$$\tilde{T} = \begin{pmatrix} \tilde{a} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & -\beta \\ -q\gamma & a \end{pmatrix} \quad (16)$$

satisfy the defining relations of the algebra $O(\widetilde{GL}_{q^2}(2))$ and thus $O(\widetilde{GL}_{q^2}(2))$ is the opposite algebra of $O(\widetilde{GL}_q(2))$.

Proof. The use of relations (6)-(8) imply

$$\begin{aligned} \tilde{a}\tilde{\beta} &= \tilde{\beta}\tilde{a}, & \tilde{\beta}\tilde{\gamma} &= \tilde{\gamma}\tilde{\beta}, & \tilde{\beta}\tilde{d} &= \tilde{d}\tilde{\beta}, \\ \tilde{a}\tilde{\gamma} &= q^2\tilde{\gamma}\tilde{a}, & \tilde{\gamma}\tilde{d} &= q^2\tilde{d}\tilde{\gamma}, \\ \tilde{a}\tilde{d} &= \tilde{d}\tilde{a} + (1 - q^2)\tilde{\beta}\tilde{\gamma}, \end{aligned}$$

which are the defining relations of the algebra $O(\widetilde{GL}_{q^2}(2))$. The second claim follows from the fact that $q^3 = 1$. \square

Proposition 3.7 *The bialgebra $O(\widetilde{GL}_q(2))$ is a Z_3 -graded Hopf algebra. The antipode S of $O(\widetilde{GL}_q(2))$ is given by*

$$S(a) = d D_q^{-1}, \quad S(\beta) = -\beta D_q^{-1}, \quad S(\gamma) = -q\gamma D_q^{-1}, \quad S(d) = a D_q^{-1}. \quad (17)$$

Proof. By Lemma 3.6, there exists an algebra anti-homomorphism S from $O(\widetilde{GL}_q(2))$ to $O(\widetilde{GL}_{q^2}(2))$ such that $S(a) = \tilde{a}$, etc. To prove that S is an antipode for $O(\widetilde{GL}_q(2))$, we have to check the antipode axiom

$$m \circ (S \otimes \text{id}) \circ \Delta = \epsilon = m \circ (\text{id} \otimes S) \circ \Delta \quad (18)$$

for the generators. To check the axiom (18) for the generators is equivalent to verify the following matrix equality

$$T\tilde{T}D_q = \epsilon(T) = \tilde{T}TD_q$$

which follows from $D_q = ad - q\beta\gamma$ in $O(\widetilde{GL}_q(2))$ with $S(T) = D_q^{-1}\tilde{T} = T^{-1}$. The details can be checked easily. \square

Definition 3.8 *The Z_3 -graded Hopf algebra $O(\widetilde{GL}_q(2))$ is called the coordinate algebra of the Z_3 -graded (quantum) group $\widetilde{GL}_q(2)$.*

3.4 Coactions on the Z_3 -graded exterior plane

In bialgebra terminology, the second suggestion of Proposition 3.2 yields the following.

Proposition 3.9 *The algebra $O(\tilde{\mathbb{R}}_q^{0|2})$ is a left and right comodule algebra of the bialgebra $O(\tilde{M}_q(2))$ with left coaction δ_L and right coaction δ_R such that*

$$\delta_L(\theta) = a \otimes \theta + \beta \otimes \varphi, \quad \delta_L(\varphi) = \gamma \otimes \theta + d \otimes \varphi, \quad (19)$$

$$\delta_R(\theta) = \theta \otimes a + \varphi \otimes \gamma, \quad \delta_R(\varphi) = \theta \otimes \beta + \varphi \otimes d. \quad (20)$$

Proof. It is not difficult to verify that (19) and (20) define algebra homomorphisms δ_L from $O(\tilde{\mathbb{R}}_q^{0|2})$ to $O(\tilde{M}_q(2)) \otimes O(\tilde{\mathbb{R}}_q^{0|2})$ and δ_R from $O(\tilde{\mathbb{R}}_q^{0|2})$ to $O(\tilde{\mathbb{R}}_q^{0|2}) \otimes O(\tilde{M}_q(2))$, respectively. It remains to be checked that δ_L and δ_R are coactions, i.e., the conditions

$$(\Delta \otimes \text{id}) \circ \delta_L = (\text{id} \otimes \delta_L) \circ \delta_L, \quad m \circ (\epsilon \otimes \text{id}) \circ \delta_L = \text{id} \quad (21)$$

and

$$(\text{id} \otimes \Delta) \circ \delta_R = (\delta_R \otimes \text{id}) \circ \delta_R, \quad m \circ (\text{id} \otimes \epsilon) \circ \delta_R = \text{id} \quad (22)$$

are satisfied. For examples,

$$\begin{aligned}
(\Delta \otimes \text{id})\delta_L(\theta) &= (\Delta \otimes \text{id})(a \otimes \theta + \beta \otimes \varphi) \\
&= (a \otimes a + \beta \otimes \gamma) \otimes \theta + (a \otimes \beta + \beta \otimes d) \otimes \varphi \\
&= a \otimes (a \otimes \theta + \beta \otimes \varphi) + \beta \otimes (\gamma \otimes \theta + d \otimes \varphi) \\
&= a \otimes \delta_L(\theta) + \beta \otimes \delta_L(\varphi) \\
&= (\text{id} \otimes \delta_L)\delta_L(\theta)
\end{aligned}$$

and

$$\begin{aligned}
m \circ (\epsilon \otimes \text{id})\delta_L(\theta) &= m(\epsilon \otimes \text{id})(a \otimes \theta + \beta \otimes \varphi) \\
&= m(1 \otimes \theta + 0 \otimes \varphi) \\
&= \theta
\end{aligned}$$

as expected. \square

Remark 4. In fact, there exists a left coaction of $O(\tilde{\mathbb{R}}_q^{*0|2})$ on the plane $\tilde{\mathbb{R}}_q^{*0|2}$, called a left comodule- $O(\tilde{\mathbb{R}}_q^{*0|2})$ satisfying the conditions (21).

Remark 5. An easy computation shows that the ideal $(\vartheta := \theta\varphi - q^2\theta\varphi)$ of $\tilde{\mathbb{R}}_q^{0|2}$ is a subcomodule of $\tilde{\mathbb{R}}_q^{0|2}$. The proof is immediate: Indeed, since δ_L is an algebra map, it is only necessary to show that $\delta_L(\vartheta) = D_q \otimes \vartheta$. Using relations (6)-(8) with (9) we get

$$\begin{aligned}
\delta_L(\vartheta) &= \delta_L(\theta)\delta_L(\varphi) - q^2\delta_L(\varphi)\delta_L(\theta) \\
&= qa\gamma \otimes \theta^2 + ad \otimes \theta\varphi + q^2\beta\gamma \otimes \varphi\theta + \beta d \otimes \varphi^2 - q^2\gamma a \otimes \theta^2 \\
&\quad - q\gamma\beta \otimes \theta\varphi - q^2da \otimes \varphi\theta - d\beta \otimes \varphi^2 \\
&= (ad - q\beta\gamma) \otimes \theta\varphi - q^2(da - \beta\gamma) \otimes \varphi\theta = D_q \otimes \vartheta
\end{aligned}$$

as expected. \square

3.5 The Hopf algebra $O(\widetilde{SL}_q(2))$

We know that, since the determinant D_q is group-like, the two-sided ideal $\langle D_q - 1 \rangle$ generated by the element $D_q - 1$ is a biideal of $O(\widetilde{M}_q(2))$. So the quotient $O(\widetilde{SL}_q(2)) := O(\widetilde{M}_q(2))/\langle D_q - 1 \rangle$ is a bialgebra.

Proposition 3.10 *There exists a Hopf \star -algebra structures on the Hopf algebra $O(\widetilde{SL}_q(2))$ such that*

$$a^\star = a, \quad \beta^\star = \beta, \quad \gamma^\star = q\gamma, \quad d^\star = d. \quad (23)$$

4 Z_3 -graded quantum algebra of $\widetilde{GL}_q(2)$

In this section, using the method of [8], we give an R -matrix formulation for the Z_3 -graded quantum group $\widetilde{GL}_q(2)$ and obtain a Z_3 -graded universal enveloping algebra $U_q(\widetilde{gl}(2))$.

4.1 The FRT construction for $\widetilde{GL}_q(2)$

The R -matrix formulation (the FRT-relation $\hat{R}T_1T_2 = T_1T_2\hat{R}$) for the quantum matrix groups [8] can be considered as a compact matrix form of the commutation relations between the generators of an associative algebra.

The formulation for the Z_3 -graded quantum group $\widetilde{GL}_q(2)$ has the same form, but matrix tensor product includes additional q -factors related to Z_3 -grading. Two matrices A, B ($\tau(A_{ij}) = \tau(i) + \tau(j)$) are multiplied according to the rule

$$(A \otimes B)_{ij,kl} = q^{\tau(j)(\tau(i)+\tau(k))} A_{ik} B_{jl}. \quad (24)$$

Due to this prescription, $T_2 = I \otimes T$ has the same block-diagonal form as in the standard (ungraded) case while $T_1 = T \otimes I$ includes the additional factors q for graded elements standing at some of odd rows of blocks. For the Z_3 -graded quantum group $\widetilde{GL}_q(2)$ the R -matrix satisfying the Z_3 -graded Yang-Baxter equation has in the form

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^2 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = (\hat{R}_{ij}^{kl}) \quad (25)$$

where $\hat{R} = \underline{P}R$ and \underline{P} denotes the Z_3 -graded permutation operator defined by $\underline{P}(a \otimes b) = q^{\tau(a)\tau(b)} b \otimes a$ on homogeneous elements. A simple calculation shows that this operator represents the 3rd-root of the permutation operator P with action $P(a \otimes b) = b \otimes a$.

The condition for the matrices to belong to the Z_3 -graded quantum group $\widetilde{GL}_q(2)$ is given below, but it will not be proved here.

Proposition 4.1 *A 2×2 -matrix T is a Z_3 -graded quantum matrix if and only if*

$$\hat{R}T_1T_2 = T_1T_2\hat{R} \quad (26)$$

where matrix elements of T are Z_3 -graded.

4.2 A Z_3 -graded universal enveloping algebra $U_q(\widetilde{gl}(2))$

The Z_3 -graded quantum algebra of $\widetilde{GL}_q(2)$ can be analogous construction to approach of the Leningrad school. The Z_3 -graded quantum algebra of $\widetilde{GL}_q(2)$ has four generators: U and V are of degree 0, X_- and X_+ are of degrees 1 and 2, respectively.

Proposition 4.2 *The generators of the Z_3 -graded quantum algebra satisfy the following relations*

$$UV = VU, \quad UX_{\pm} = q^{\pm 2} X_{\pm} U, \quad VX_{\pm} = q^{\mp 2} X_{\pm} V, \quad (27)$$

$$X_+ X_- - X_- X_+ = \frac{UV^{-1} - VU^{-1}}{q^2 - q} \quad (28)$$

proof. The generators U, V, X_{\pm} can be written in two 2x2 matrix as follows

$$L^+ = \begin{pmatrix} U & \lambda X_+ \\ 0 & V \end{pmatrix}, \quad L^- = \begin{pmatrix} U^{-1} & 0 \\ \lambda X_- & V^{-1} \end{pmatrix} \quad (29)$$

where $\lambda = q - q^2$. The matrices L^{\pm} satisfy the following relations

$$R^+ L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R^+, \quad (30)$$

where the matrix R^+ is defined by $R^+ = \underline{P} R \underline{P}$. The relations (27) follow from the relations (30). To obtain the relation (28) we use the relation

$$R^+ L_1^- L_2^+ = L_2^+ L_1^- R^+. \quad (31)$$

Proposition 4.3 *The coproduct of the generators is given by*

$$\Delta(L^{\pm}) = L^{\pm} \dot{\otimes} L^{\pm}. \quad (32)$$

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